Solution to Assignment 12

Section 9.2

1. (a) Let $x_n := \frac{1}{(n+1)(n+2)}$. We have $|x_{n+1}/x_n| = (n+1)/(n+3) = 1 - 2/(n+3)$, so

$$\lim_{n\to\infty} n\left(1 - \frac{|x_{n+1}|}{|x_n|}\right) = 2 > 1 \ .$$

By the limit version of Raabe's Test, the series converges absolutely.

An alternate method. Observe that $x_n \leq 1/n^2$ and $\sum_n n^{-2} < \infty$. By Comparison test, $\sum x_n$ converges absolutely since each x_n is positive.

(c) Since $\lim_{n\to\infty} 2^{-1/n} = 2^0 = 1 \neq 0$. $\sum 2^{-1/n}$ diverges.

2. (c) Since

$$\frac{|x_{n+1}|}{|x_n|} = \frac{(n+1)!}{(n+1)^{n+1}} \times \frac{n^n}{n!} = \left(1 + \frac{1}{n}\right)^{-n} \to e^{-1} < 1 \ ,$$

as $n \to \infty$, we apply the limit version of Ratio Test to conclude absolute convergence.

An alternate method. $x_n = \dfrac{n!}{n^n} = \dfrac{n(n-1)\cdots 21}{nn\cdots nn} \leq \dfrac{2}{n^2}$. Therefore by comparison test, the series diverges.

(d) Denote $x_n := (-1)^n \dfrac{n}{n+1}$. Then $\lim x_{2n} = 1$ and $\lim x_{2n-1} = -1$. Since there is no limit (let alone tending to 0), $\sum x_n$ diverges.

3. (b) We have

$$((\log n)^{-n})^{1/n} = 1/\log n \to 0$$

as $n \to \infty$, by the limit version of Root Test we conclude that the convergence is absolute.

(c) See if we can find some n_0 such that

$$(\log n)^{-\log n} \leq n^{-2}$$

for $n \geq n_0$. Taking log both sides to get

$$(-\log n)\log\log n \leq -2\log n \ ,$$

which is

$$-\log\log n \leq -2,$$

and it holds for some n_0. Hence the series is convergent by Comparison Test.

(d) Using $\log n \le n$, we have

$$\frac{1}{(\log n)^{\log\log n/n}} \ge \frac{1}{n^{\log\log n/n}} \ .$$

Using $\log\log n \le \log n \le n$ we further have

$$\frac{1}{n^{\log\log n/n}} \ge \frac{1}{n^{n/n}} = \frac{1}{n} \ .$$

As $\sum_n n^{-1} = \infty$, by comparison test

$$\sum_n \frac{1}{(\log n)^{\log\log n/n}} \ge \sum_n \frac{1}{n} = \infty \ .$$

That is, this series is divergent.

(e) Use Integral Test to the function $f(x) = \log\log x$ to conclude divergence.

4. (b) Denote $x_n := n^n e^{-n}$. We have $|x_n|^{1/n} = n/e \to \infty$ as $n \to \infty$. By the limit version of Root Test we have divergence. Alternatively,

$$\left|\frac{x_{n+1}}{x_n}\right| = \frac{n+1}{e}\left(1+\frac{1}{n}\right)^n \ge \frac{2(n+1)}{e} \to \infty \ .$$

By the limit version of Ratio Test, $\sum x_n$ diverges.

(c) $a_n = e^{-\log n} = 1/n$ is divergent.

(d) We use Ratio Test. We have

$$\frac{|x_{n+1}|}{|x_n|} = \frac{\log(n+1)}{\log n}\frac{1}{e^{\sqrt{n}+\sqrt{n+1}}} \to 0 \ ,$$

as $n \to \infty$. By the limit version of Ratio Test, this series is absolutely convergent.

(e) $a_n = n!e^{-n}$. By Ratio Test in Limit Form, $a_{n+1}/a_n = e/(n+1) \to 0$, hence it is convergent.

6. Define $f(x) := (ax+b)^{-p}$. Then $f'(x) := -ap(ax+b)^{-p-1} < 0$, for $x \ge 1$. Moreover,

$$\int_1^R f = \int_1^R \frac{dx}{(ax+b)^p} = \begin{cases} \left.\dfrac{(ax+b)^{1-p}}{a(1-p)}\right|_1^R, & \text{for } p \ne 1 \\ \ln(ax+b)\big|_1^R, & \text{for } p = 1 \end{cases}$$

$$= \begin{cases} \dfrac{1}{a(1-p)}\left(\dfrac{1}{(aR+b)^{p-1}} - \dfrac{1}{(a+b)^{p-1}}\right), & \text{for } p \ne 1 \\ \ln(aR+b) - \ln(a+b), & \text{for } p = 1 \end{cases}$$

If $p > 1$, then $\displaystyle\lim_{R\to\infty}\int_1^R f = \frac{(a+b)^{1-p}}{a(p-1)}$, by integral test, $\sum (an+b)^{-p} < \infty$.

If $p \le 1$, then $\displaystyle\int_1^R f$ diverges as $R \to \infty$, by integral test, $\sum (an+b)^{-p}$ diverges.

7. (a) Denote $x_n := \dfrac{n!}{3 \cdot 5 \cdot 7 \cdots (2n+1)}$. Then $\left|\dfrac{x_{n+1}}{x_n}\right| = \dfrac{n+1}{2n+3} \to \dfrac{1}{2} < 1$.

   By the limit form of Ratio Test, $\sum x_n$ converges absolutely.

   (b) $a_n = (n!)^2/(2n)!$. As $a_{n+1}/a_n = (n+1)^2/(2n+1)(2n+2) \to 1/4$, it is convergent by Ratio Test (Limit Form).

   (c) Denote $x_n := \dfrac{2 \cdot 4 \cdots (2n)}{3 \cdot 5 \cdots (2n+1)}$. Then $\left|\dfrac{x_{n+1}}{x_n}\right| = \dfrac{2n+2}{2n+3} = 1 - \dfrac{1}{2n+3}$. Therefore,

$$\lim_{n \to \infty} n\left(1 - \left(1 - \frac{1}{2n+3}\right)\right) = \frac{1}{2} \ ,$$

   which implies that the series diverges by the limit form of Raabe's Test.

8. Note that this series is a rearrangement of $a, a^2, \ldots, a^{n-1}, a^n, \ldots$, which we already know is absolutely convergent.

   Root test:

$$|x_n|^{1/n} = \begin{cases} a^{(n-1)/n}, & \text{n = 2k;} \\ a^{n/(n-1)}, & \text{n = 2k-1.} \end{cases}$$

   In both cases $|x_n|^{1/n} < 1$. By root test, the infinite series is convergent.

   Ratio test:

$$\frac{x_{n+1}}{x_n} = 1/a > 1 \quad \forall n = 2k+1, k \in \mathbb{N}$$

   and

$$\frac{x_{n+1}}{x_n} = a^2 < 1 \quad \forall n = 2k, k \in \mathbb{N}$$

   We can't use ratio test to judge if this series is convergent.

17. Applying the limit version of Raabe's Test

$$n\left(1 - \frac{x_{n+1}}{x_n}\right) = \frac{n(q-p)}{q+n+1} \to q - p \ , \qquad \text{as } n \to \infty \ .$$

   Therefore, we have convergence if $q - p > 1$ and divergence if $q - p < 1$. When $q = p + 1$, $\sum_n x_n = \sum_n 1/(q+n) = \infty$, so we have divergence in this case.

19. We adopt the notation in the question. Since $b_1 = \sqrt{A} - \sqrt{A_1}$ and $b_n = \sqrt{A - A_{n-1}} - \sqrt{A - A_n} > 0$,

$$\sum_{k=1}^{N} b_k = \sqrt{A} - \sqrt{A - A_N} \to \sqrt{A} \text{ as } N \to \infty.$$

   Hence the series converges. Now, let us verify that $\lim_{n\to\infty} a_n/b_n = 0$. For $n > 1$,

$$b_n = \sqrt{A - A_{n-1}} - \sqrt{A - A_n} = \frac{A_n - A_{n-1}}{\sqrt{A - A_{n-1}} + \sqrt{A - A_n}} = \frac{a_n}{\sqrt{A - A_{n-1}} + \sqrt{A - A_n}}.$$

   Using the fact that $\lim_{n\to\infty} A_n = A$, we conclude that

$$\frac{a_n}{b_n} = \sqrt{A - A_{n-1}} + \sqrt{A - A_n} \to 0 \text{ as } n \to \infty.$$

20. Let $b_n = a_n/\sqrt{A_n}$ where A_n is the nth partial sum of $\sum a_n$. It is clear that

$$\lim_n (b_n/a_n) = \lim_n 1/\sqrt{A_n} = 0$$

since $\sum a_n$ is divergent. Now we prove $\sum b_n$ is also divergent.

$$\sum b_n \geq \sum_{n=1}^{M} b_n \geq \sum_{n=1}^{M} a_n/\sqrt{A_M} = \sqrt{A_M} \quad \forall M \in \mathbb{N}$$

Letting $M \to \infty$, we have the desired conclusion.